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Review

General principles of chaotic dynamics

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The following survey of the fundamental principles of chaos is intended to provide the reader of our *spotlight issue* with an overview of some of the most essential facts required for understanding chaotic dynamics. It is a great honour that leading scientists from the field of chaos in the cardiovascular system have agreed to provide focused reviews on several specific topics. The interested reader may also wish to refer to excellent previous reviews dealing with the microvasculature [12,15], heart rate [6,9], renal haemodynamics [17] and chaos in general [3,5,11,13,37,40]. For reviews of chaos theory in the physical literature [1,7,8,29] or textbooks on chaos, the reader may refer to various other sources [16,18,20,26,28,30,32, 34,39].

1. History

Beginning with the 17th century, the understanding of nature advanced through its first significant change in paradigm in modern times. Johannes KEPLER (1571–1630) paved the way for the secularisation of natural sciences. He tried to prove the harmony of the structure of the solar system.

The general principles of mechanics were established in the works of Isaac NEWTON (1643–1727), *Philosophiae naturalis principia mathematica* and Gottfried LEIBNITZ (1646–1716) *natura non facit saltus*: the past and future of the material world was particularised. Assuming that all laws of nature are "causal", Pierre LAPLACE (1749–1827) introduced the ficticious figure of a demon, who, owing to his computational capacity, was able to predict the future state of the universe. The laws of motion were manifested in well-behaving solutions: A pendulum oscillates in a harmonic manner and the planets' orbits around the sun consist of circles and ellipses.

As Ludwig BOLTZMANN (1844–1906) developed the field of statistical thermodynamics, it became clear that there was a limit to mechanics and the classical description of nature. WEIERSTRASS formulated a tricky problem, which was later pursued by the Swedish King Oscar II: Is the solar system stable or unstable? This prize question should clarify the potential effects of resonances in the solar system: do they cumulate such that the planets detach themselves some time in the future or are they negligible? The prize was won by Henry POINCARÉ (1854–1912), who proved that the so-called "three-body problem" cannot in general be solved analytically. Moreover, he showed that there are stable and unstable types of orbits and that sometimes even a tiny disturbance in the system can bring about a change in the nature of the orbit.

Poincaré particularly examined the issue of predictability. On the one hand, the systems are deterministic, but on the other, the strong priciple of causality is violated: similar causes do not lead to similar effects. All these systems and problems are non-linear but can be described by differential equations (i.e., they are deterministic). Calculus can prove the existence of a solution, but cannot provide the solution itself: the systems cannot be solved analytically. There is no formula that relates the state of such a system at a given time to the state at a future time. Poincaré developed mathematical tools for the analysis of such problems (e.g., the so-called Poincaré surface of section).

It emerged that a vast number of laws of nature are indeed non-linear: in the description by differential equations, the variables and their derivatives with respect to

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time are coupled in a non-linear manner (i.e., squares of variables, sines, cosines etc.). Remarkably there was another revolution in physics this century: the development of quantum mechanics. Quantum-mechanical descriptions of nature are in terms of probability statements, which were unfamiliar to the physicist community at that time. This is one of the reasons why the investigation of non-linear systems was not pursued for some time.

In 1961, Edward LORENZ again encountered the phenomenon of "sensitive dependence on initial conditions". He worked on a very simple mathematical model of the weather, which he tried to solve with the aid of a computer. He recognised that starting the computer program with slightly different initial conditions (temperature, atmospheric pressure etc.) than the previous run eventually resulted in totally different weather conditions. This was clear evidence for the violation of the strong principle of causality. Today this fact is paraphrased by the term *butterfly effect*.

The Lorenz model has been extensively studied over the past several years and since the beginning of the 1980s a keen interest in investigating non-linear systems was observed. In 1975 Li and Yorke introduced the term "chaos" for behaviour similar to that observed by Lorenz; this field is also referred to as "Dynamical Systems Theory" or "Chaos Theory" [5].

2. Phase space and description of deterministic systems

In general, the description of a deterministic system is concerned with the properties of its phase space or state space. A phase space is a coordinate system whose axes are defined by the independent variables of the system under study. For example, a swinging pendulum can be fully described by the angle and the associated angular velocity. Thus, the phase space of the pendulum is said to be two-dimensional. But in real life, most systems of interest are very high-dimensional. For example, a mol of gas consisting of approx. 10^{23} particles has a phase space which is 10^{24}-dimensional (three state coordinates for each particle and three coordinates for velocity or momentum). To overcome this problem, a statistical description for systems with many degrees of freedom (thermodynamics) was introduced. In analogy to this example, biological systems may have an almost infinite number of degrees of freedom!

When examining systems with an almost infinite number of degrees of freedom, it is often revealed that in spite of possessing a vast number of independent variables, only a few are necessary to describe their behaviour. Measuring one variable of such a system often shows a principle mode or frequency (e.g., the cardiac pacemaker).

Thus, relevant information concerning the whole system (organism) may be extracted from the observation of a single variable (heart rate, blood pressure). This signal contains information about the complete system, since all degrees of freedom, controllers, or regulating mechanisms are intertwined. This also holds true for linear systems, but in the case of non-linear systems there are interesting features that cannot be observed in linear systems (i.e., non-integer dimensions of the attractors, positive Lyapunov exponents etc.). Reducing the entire system's dynamics to a "simpler form" of behaviour makes it possible to investigate some general but important aspects of the overall system behaviour.

3. Continuous-time systems and discrete-time systems

Continuous-time dynamical systems are defined by

$$\dot{x} = f(x) \text{ and } \dot{x} = f(x,t), \tag{1}$$

where in the first equation the system is called *autonomous* (the vector field or *flow f* does not depend on time), and in the latter a one *non-autonomous system* (where the vector field depends on time). Eq. (1) are partial differential equations that relate the variables $\mathbf{x}$ to their derivatives with respect to time, $\dot{\mathbf{x}}$. Often, non-autonomous systems are time-periodic (e.g., systems driven by an external periodic force). If we try to model a system where time has a discrete character, Eq. (1) have to be replaced by:

$$x_{k+1} = P(x_k). \tag{2}$$

P maps the state $\mathbf{x}_k$ to the next state $\mathbf{x}_{k+1}$. Starting with an initial state $\mathbf{x}_0$, application of the *map P* yields $\mathbf{x}_1$, a further application gives $\mathbf{x}_2$, and so on. This results in a series of states $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2, \cdots$. An example of such an iteration is the population dynamics of a swarm of mayflies, where $\mathbf{x}_i$ is the number of flies at day i and $\mathbf{x}_{i+1}$ the population at the next day $i+1$.

At each instant, the representation by a point in phase space is unique. Continuous-time systems move along a continuous trajectory in phase space, whereas time-discrete systems are characterised by discrete points in phase space. With time, the system is drawn towards an *attractor* on which the system moves or comes to rest. There are several types of attractors: fixed points, limit cycles, tori and strange attractors (see glossary).

4. Delay differential equations

Delay differential equations (DDE) are widespread in biological sciences, and a large number of control systems can be modelled by these relations. In general, a DDE relates a state at time t to a state at some former time $t-\tau$ [21]:

$$\dot{x} = f(x(t), x(t-\tau)). \tag{3}$$

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Since the number of states x_l in the interval $l:[t-\tau,t]$ is infinite, the dynamical system (3) is infinite-dimensional as well. The time-evolution of (3) is imposed by the (very often) non-linear feedback $f(x)$ which includes a time delay. This non-linear feedback is often sigmoidal in shape and can be modelled by the Hill function:

$$y = y_{\max}\frac{x^m(t-\tau)}{a^m + x^m(t-\tau)},$$

whereby parameter a and the Hill coefficient **m** are determined from experimental data [27]. The solutions of (3) proved to be periodic trajectories and strange attractors, depending on the parameter values [21].

5. Graphical tools for visualisation of phase space dynamics

In this section, we will discuss some graphical tools suitable for the characterisation of attractors, which are explained by the continuous-time van der Pol oscillator.

The van der Pol oscillator is a continuous-time system, which is described by the non-autonomous differential equation:

$$\left.\begin{aligned} \dot{x} &= y \\ \dot{y} &= a(1-x^2)y - x^3 + k\cos(z) \\ \dot{z} &= \Omega \end{aligned}\right\} \qquad (4)$$

where k determines the coupling of an external driving force to the oscillator. Depending on the magnitude of the coupling constant k, the attractor takes on different shapes. In Fig. 1, three solutions of Eq. (4) are shown, depending on the parameter k. When $k=1$ the system is defined by a torus in phase space. A section of the torus (Poincaré section, see below) at $y=0$ shows that the attractor is the surface of the torus. As parameter k increases, the solution results in a closed periodic trajectory. The Poincaré surface of section is therefore determined by one or several distinct points (Fig. 1 B). $k=17$ defines a strange attractor. Now the trajectory seems irregular, but the surface of section reveals a very complicated structure far from randomness.

Responsible for the variety of solutions, including the chaotic ones, are the non-linear terms in the equation of motion. For the van der Pol oscillator, this is guaranteed by the second equation of (4). But this alone does not account for chaotic behaviour. Whether a non-linear system moves towards a fixed point or behaves periodically or chaotically, depends on the value of one or more control parameters (a, k, and Ω in our example). In Fig. 1, the strength of the coupling of the oscillator to the external force, expressed by k, is altered. With differing values of the control parameter, the system changes from quasi-periodic to periodic motion and finally to chaotic behaviour. This feature is termed *route to chaos* and will be explained by the logistic equation below.

Phase portrait. In order to visualise the dynamics of a system, a value $x(t)$ is plotted against the value $x(t-\tau)$, where τ is a time delay (Fig. 2A,B). Repeating this procedure results in a phase portrait (i.e., a projection of the phase space trajectory or attractor onto two dimensions). Selecting the proper delay time τ can be done in several ways, and there exist several suggestions for it, whereby the two values used for reconstruction must be uncorrelated. Thus, a choice of τ should be made in terms of a decorrelation time of the signal. One possibility is to calculate the autocorrelation function of the time series x and to choose τ as the time where the autocorrelation function has its first zero value.

Return map. In contrast to a phase portrait, the return map is a discrete description of the underlying dynamics. From an experiment, a series of return points of the oscillator is obtained. These values are used to "recon-

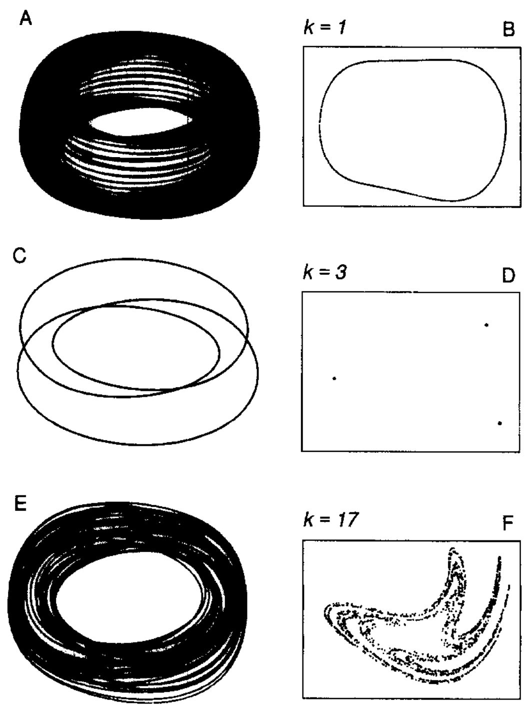

Fig. 1. Three solutions for the van der Pol oscillator with different parameter values k quantifying the strength to a periodic external force (Eq. (4)). The left panel shows the attractors for different parameter values, whereas the right panel shows the associated Poincaré sections. For $k=1$ the attractor is the (two-dimensional) surface of a torus in 3-dimensional space. The motion is characterised by two incommensurable frequencies (A). The Poincaré surface of section is a closed curve (B). For $k=3$ the trajectory closes after orbiting the center of the coordinate system 3 times. Therefore, the behaviour is characterised by a fundamental frequency and several integer multiple frequencies (harmonics) (C). The Poincaré section simply consists of 3 points indicating the breakthroughs (D). In the case $k=17$, however, the motion is purely chaotic. The motion seems irregular (E), but the surface of section shows that there is order in chaos (F). The attractor (E) is referred to as a *strange attractor*.

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struct an iteration'': a retrogressive graphical iteration is performed by plotting the return point i (e.g., the minima of the signal x) against the return point $i-1$ (Fig. 2C,D). Thus, the experimental time series is interpreted as an iteration. The return map reveals a functional relation of the successive minima. This correlation is due to the determinism ruling the system. The return map shows interrelations concerning the signal, which are not visible to the naked eye.

Poincaré surface of section. A further way of determining a possible structure of a system is the investigation of an attractor when it penetrates a plane in phase space. A surface or plane is fixed and all transitions of the multi-dimensional attractor from one side of the plane to the other are sampled. This method is termed ''Poincaré section''. With regard to the harmonically forced oscillator, defining the section at a fixed phase ϕ of the external force is evident, meaning that the whole process is visualised stroboscopically synchronised with the stimulus. In the case of a periodic motion, it is clear that the system leaves

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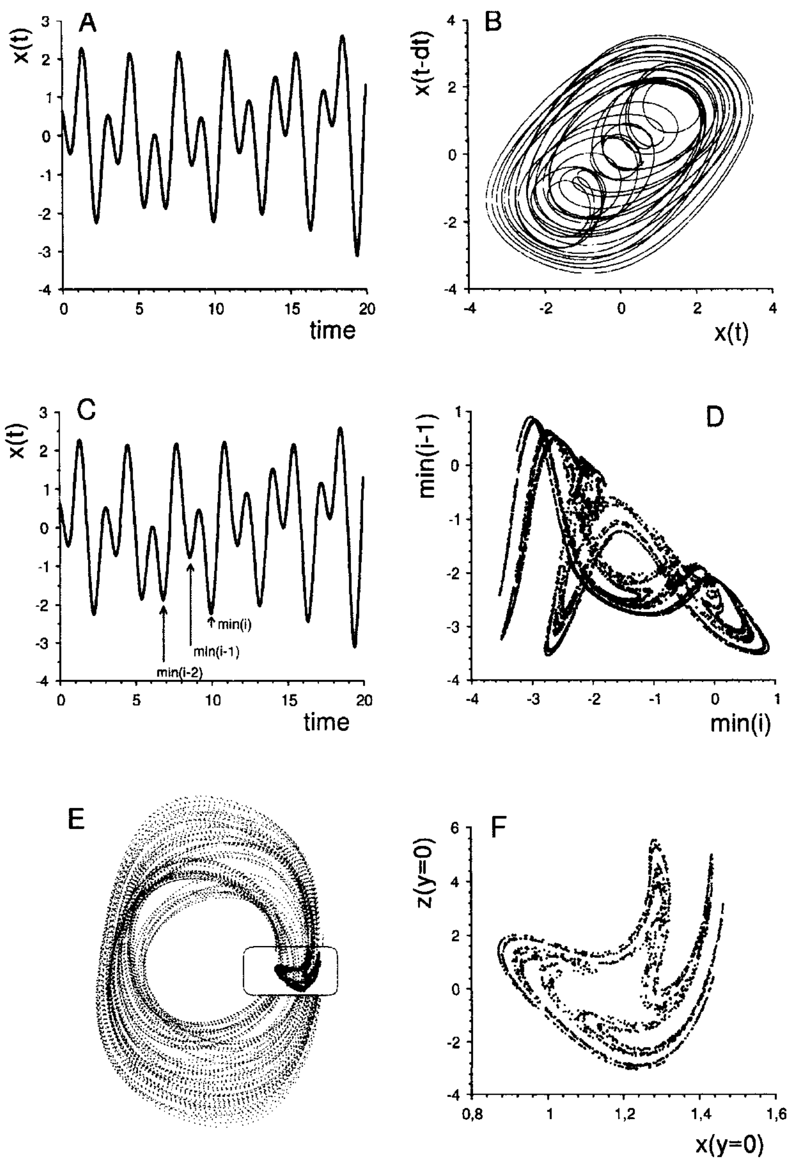

Fig. 2. Graphical analysis of the van der Pol oscillator, where the control parameter k was 17 (chaos). A *phase portrait* in two dimensions is generated by plotting one data point $x(t)$ (A) against another point $x(t-\delta t)$, where the two values are separated by a fixed time lag δt (B). This procedure gives an impression of the phase space structure of the attractor. Phase portraits can be performed in more than two dimensions (*embedding dimension*). If the embedding dimension exceeds twice the fractal dimension, one gets a strange attractor which looks different from the original one but has the same fractal dimension and Lyapunov exponents. A *return map* is generated by plotting one return value of the time series against the previous one (C). Repeating this procedure gives an indication to the order within chaos (D). In (E) a *Poincaré section* was performed, collecting all points where the trajectoy passes from one side of the plane to the other (the surface of the section was localised at $y=0$). Inset (F) shows the points where the trajectory penetrates the surface. Like the return map, the Poincaré section shows that there is order in chaos.

only one or a limited number of points on the surface, provided that the relation of the frequencies of the pendulum and that of the external force is rational. Accordingly, this number is equal to the number of periods of the external force, after which the trajectory closes. When the two frequencies are incommensurable, the motion is still periodic, but the orbit never closes. Now the system moves on the surface of a torus, which is the attractor for such a quasi-periodic system (Fig. 1A). In contrast to periodic motion, there is no closed curve when the system is chaotic. Evidently, the points are not scattered on the surface but reveal a certain structure. Depending on the control parameter k, this structure is more or less complicated. From the complex morphology of Fig. 2F, one may conclude that the trajectory moves along a structure in three-dimensional phase space, which corresponds to a multiple folded area. As opposed to the simple one- and two-dimensional attractors of periodic processes, this is called a strange attractor.

6. Bifurcations, self-similarity, and universality

In 1845, the biologist Verhulst introduced the *logistic equation*:

$$X_{i+1} = kX_i(1 - X_i), \qquad (5)$$

in order to model animal population, whereby X is the variable (the population) and k a control parameter [3,24,25]. Proceeding from the number of animals in generation i: N_i, the number of animals born in the next generation is kN_i, whereby k is the reproduction factor. Such a population would develop exponentially, thus a factor $(N_{max} - N_{old})$ is introduced to stem the growth. (N_{max} may be the maximum number of animals for which food and space are available). The larger the current population, the smaller this factor is. Finally, Eq. (5) is normalised to 1.

Depending on the parameter k, three different types of behaviour can be observed when Eq. (5) is iterated. After a transition period, the series X_i is drawn towards some value. With differing parameters k, the iteration may result in an alternation between two or more values. Finally, with characteristic parameter values k, no stable limit is reached and the series X_i seems to be irregular. A *Feigenbaum diagram* is obtained by repeated iteration of Eq. (5) with several parameters k (Fig. 3E,F). One can discern a bifurcation scenario where the control parameter responsible for the appearance of the bifurcations and chaos is represented by k. For parameter values $k < 3$, each iteration yields a single value. At $k = 3$ a period doubling occurs: for $3 < k < 3.449$ the variable X fluctuates between two values. If k is further increased, more and more bifurcations take place and after reaching a certain parameter value, the series X_i becomes chaotic. At smaller scales, the chaotic regime is interspersed with periodic windows. When such a window is magnified, a similar copy of the whole diagram appears.

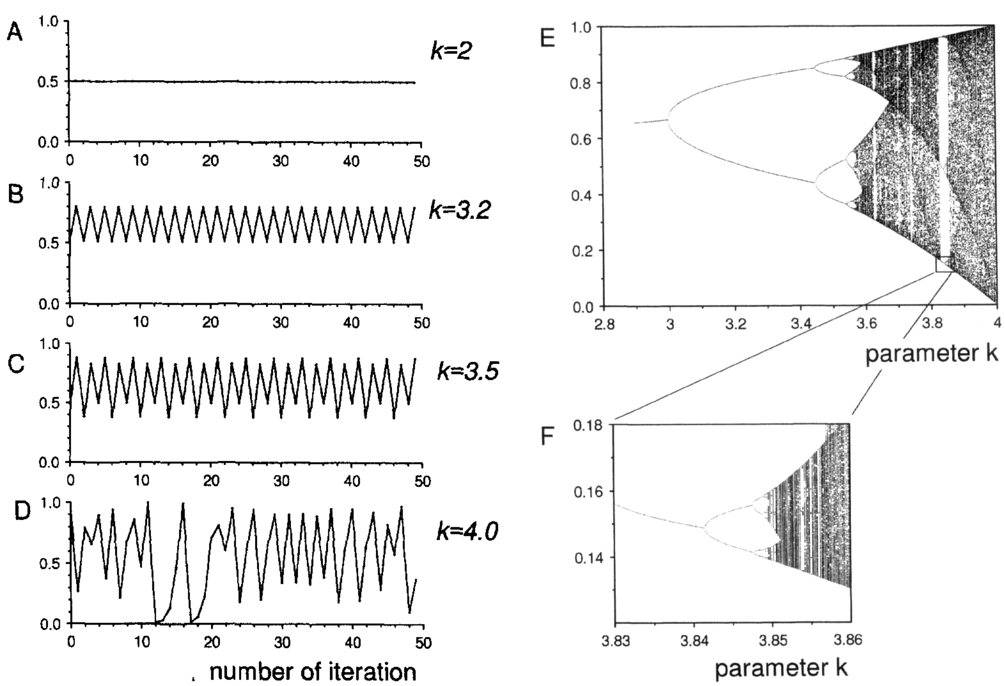

Fig. 3. The logistic Eq. (5) with different values of control parameter k: constant value (A), 2 states (B), 4 states (C), and chaos (D). The graphical presentation of the distinct states plotted against the control parameter is called a *Feigenbaum diagram* (F). The Feigenbaum diagram is self-similar: i.e., a magnification of a small part of the entire structure looks like the original. The Feigenbaum diagram can be found in a variety of different systems, this property is called *universality*.

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Due to this property, the Feigenbaum diagram is called *self-similar*, having consequences for the bifurcation points: Successive distances of two bifurcation points k_i always have the same ratio:

$$\frac{k_{i-1}-k_i}{k_i-k_{i+1}}=\delta=4.669...$$

Thus, the branching points represent a convergent series with the limit k_∞:

$$k_i=k_\infty-\text{const}\cdot\delta^{-i}\text{ and }c_\infty=3.5699...$$

The Feigenbaum diagrams constructed from other systems bear resemblance to the one obtained from the logistic equation, provided that the function f: $X_{i+1}=f(X_i)$ has a quadratic maximum. In honour of Mitchel Feigenbaum, δ was termed "Feigenbaum constant". This quantitative behaviour of several types of systems is now referred to as *universality*.

Note that a number of geometric shapes in nature are indeed self-similar: e.g., lakes, clouds, and the distribution of galaxies in outer space [23], as well as coastlines [4,19,22].

Routes to chaos. Increasing the control parameter k up to 4, we have seen that the attractor changes from stable points over periodic orbits to a strange attractor. This pathway is termed the "period-doubling" route to chaos. There are other ways from periodic motion to chaos. In Fig. 4, the attractor of the logistic equation with $k=3.82814$ exhibits a short-term chaotic behaviour "embedded" in an almost periodic motion (*intermittency*). Moreover, with parameter values a little bit smaller than 3.82814, the motion is purely periodic. For values slightly greater than 3.82814, the motion is chaotic (for this and other routes to chaos, see the reviews [2,7]).

7. Properties of strange attractors

In order to estimate fractal dimensions and Lyapunov exponents, the entire phase space of a system has to be reconstructed from a single time series. Packard et al. showed [31] that the reconstructed phase space must be at least of dimension $2d_f+1$, where d_f is the (fractal) dimension of the attractor. Parallel to the computing of the phase portrait, the components of the reconstructed phase space vectors are values of the time series x, shifted by some time lag τ:

$$\begin{aligned} v_o &= (x(0),x(\tau),x(2\tau),\ldots,x((d-1)\tau)\\ &\vdots\\ v_i &= (x(i),x(i+\tau),x(i+2\tau),\ldots,\\ &\quad x(i+(d-1)\tau). \end{aligned}$$

The result is a generalisation of the phase portrait in higher dimensions. The dimension d (= embedding dimension) of the reconstructed phase space has to be at least twice as large as the fractal dimension d_f of the attractor. Since d_f is unknown at first, a series of computations with increasing d has to be performed (see Fig. 5).

Chaotic systems are drawn towards attractors that have, in most cases, a non-integer dimension: i.e., a *fractal dimension* [23,33]. A point has a dimension of zero, a line or a curve 1, a plane is of dimension 2, and a volume can be described by three coordinates. The van der Pol oscillator with $k=1$ moves in its phase space on a torus and fills the surface densely. Therefore, the dimension of the attractor is 2. In the case of periodic motion, the dimension is 1 since the attractor is a closed curve. For the chaotic oscillator ($k=17$), the Poincaré surface consists of points which do not fill the surface densely. On the other hand, the intersection points cannot be described by one or more lines or curves. So the dimension of the Poincaré surface lies between 1 and 2, and consequently its dimension is non-integer. Since the surface of section reduces the dimension by 1, the dimension of the attractor in the complete phase space is between 2 and 3.

Box dimension. One method to compute fractal dimensions is by box-counting. The phase space is covered densely with boxes of side length l_i and the number of boxes N_i, containing parts of the trajectory, are counted. Now the boxes are scaled down and one again determines the number of boxes that contain parts of the attractor as a function of the new side length of the boxes. Plotting double logarithmically N_i as a function of $1/l_i$, one should obtain a line where the slope is equal to the fractal dimension d_f.

Correlation dimension. Computationally more efficient than box-counting is the estimation of the correlation dimension, as introduced by Grassberger and Procaccia [14]. To this end, a distance distribution function is defined, the correlation function $C(r)$. It is the number of all distances d_{ij} between each of two points x_i and x_j, which are smaller than a given r. Akin to the plane- or volume-filling with boxes, the distance distribution of self-similar structures follows a power law:

$$C(r)\sim r^{d_f}.$$

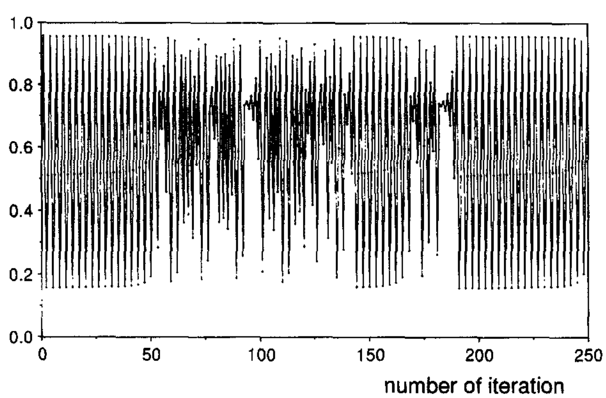

Fig. 4. A solution of the logistic equation for $k=3.82814$. Almost periodic motion is interspersed with "windows" of chaotic behaviour. This mode is termed *intermittency* and is one possible *route to chaos*.

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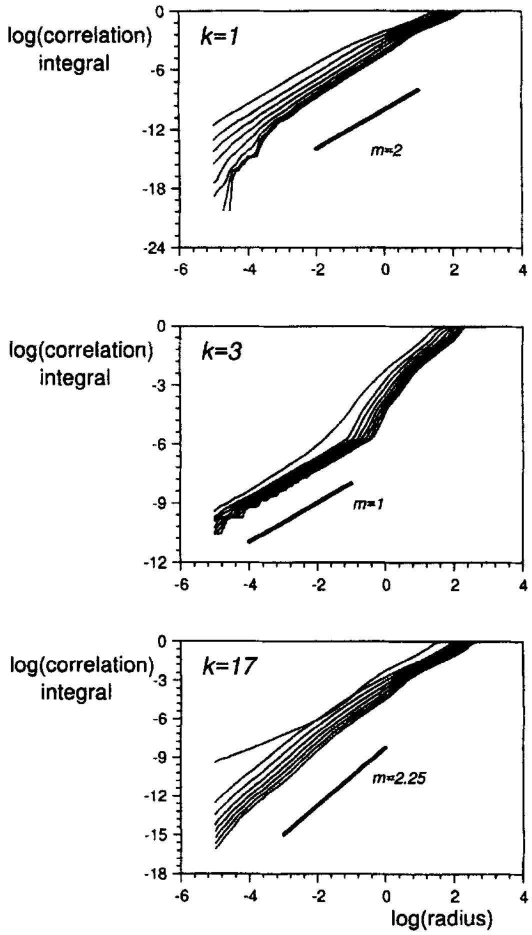

Fig. 5. Estimation of the correlation dimension of the van der Pol oscillator with different values of the control parameter. There are segments in the correlation integral, where the slope is almost constant. The slope is an estimation of the correlation dimension. In (A) $k = 1$ which is the quasi-periodic motion on the surface of a torus, therefore the correlation dimension is 2. In (B) the motion is periodic and the trajectory closes after a finite period ($k = 3$). The correlation dimension is that of a line, namely 1. When $k = 17$ the motion is chaotic and the slopes converge towards a value that lies between 2 and 3 (C). The correlation integrals were calculated with smallest embedding dimension 3 increasing in steps of 1.

In a fashion similar to box-counting, $C(r)$ is computed as a function of r and plotted double logarithmically against r. The slope of the regression line is an estimation for the fractal dimension.

Lyapunov exponents. One of the most striking properties of chaotic behaviour of deterministic systems is the divergence of nearby trajectories in phase space. Assuming that two trajectories $a_1(t)$ and $a_2(t)$ are starting with a small difference (i.e., a small distance in phase space), for the distance Δ the following relation holds:

$$\Delta(t) = |a_2(t) - a_1(t)| = |a_2(0) - a_1(0)| \cdot \exp(\lambda t) = \epsilon \cdot \exp(\lambda t).$$

The strength of divergence in phase space therefore depends on the magnitude of λ; thus, λ is a quantitative measure for chaotic behaviour.

In general, a system with an n-dimensional phase space has n Lyapunov exponents $\lambda_1, \lambda_2, ..., \lambda_n$. Each exponent gives the average rate of divergence (or convergence in the case $\lambda_i < 0$) along one axis in phase space. Whether a system is chaotic or not can be determined from the largest Lyapunov exponent alone: If it is positive, the system is chaotic (apart from stochastic systems, which may also have a positive Lyapunov exponent).

The Lyapunov exponent λ characterises the behaviour of a system: $\lambda < 0$ means that the trajectory moves towards a fixed point. $\lambda = 0$ holds for periodic systems and $\lambda > 0$ is an indication for chaotic or stochastic systems.

8. Surrogate data analysis

Due to the potential errors and restrictions of the above-mentioned techniques, this developed "check" is a significant development in chaos research [38]. The rationale behind this technique is to create an imitation of the original data set with identical linear properties as the original time series (i.e., matching means, standard deviations, power spectra, and auto-correlation functions). The difference between the two data sets is that the surrogates have shuffled up phases, thus obscuring the initial non-linear characteristics.

Several pitfalls of non-linear techniques can be ruled out by comparing these two resembling data sequences. For example, linear stochastic systems can yield false positive results with the Grassberger-Procaccia correlation dimension algorithm [14]. To preclude this, a comparison of the dimension estimates between the original data and several sets of surrogates is made. In case the measured time series is significantly different from the surrogates, one can assume that the original signal is not merely linearly correlated noise. Basically, all the methods outlined in this review (e.g., the Lyapunov exponent) can be confirmed by the surrogate method.

9. Conclusions and outlook on controlling chaos

Many hitherto inexplicable phenomena can be understood by the emerging field of non-linear dynamics and chaos. Characteristics of chaos can be obtained by the above-mentioned techniques; however, it should be stressed that these methods are of a qualitative and usually not of a quantitative nature. Chaotic behaviour typically exhibits a phase portrait with banding and forbidden zones, and shows a sensitive dependence on initial conditions, which constitutes the hallmark of chaos and is quantified by a positive Lyapunov exponent. The fractal dimension is normally low for chaotic behaviour in contrast to totally random dynamics where it is arbitrarily high. The fractal dimension can be employed as an estimate of the minimal

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number of degrees of freedom that a process obeys. Characterisation of the cardiovascular system by non-linear techniques is a black-box approach; nevertheless, the derived measures are useful for describing circulatory control as a whole. Perhaps it makes more sense to adjust body functions according to these parameters or future measures derived from non-linear dynamics rather than to fit them into our conventional ''mean value approach''. Indeed, totally new applications of non-linear dynamics may come forth, as indicated by strategies developed for controlling chaos [36].

The butterfly effect, or the extreme sensitivity to minute perturbations, could possibly be employed to stabilise dynamic behaviour and direct it to a desired state, very much like balancing a stick on one's palm. Incorporating chaos deliberately into practical systems therefore yields a maximum flexibility in their performance. Intriguingly, Garfinkel and associates recently attempted to control chaos in cardiac tissue [10] in which rhythmic action potentials were transformed into a chaotic series of events by administration of ouabain and noradrenaline. Their work may provide an early and elegant demonstration that the extraordinary sensitivity to initial conditions not only makes a system unpredictable but also makes them extremely susceptible to control. A ''proportional perturbation feedback'' method of controlling chaos was developed, which consisted of an initial learning phase and an intervention phase. During the 5–60 second learning phase, a computer constructed a Poincaré map of the action potential intervals. In the cases where chaos was successfully controlled, the chaotic pattern of the arrhythmia was converted to a low-order periodic pattern by single intermittent stimuli determined by computer. If this concept proves to be true, it may be a first step of a new generation of smart pacemakers used to restore cardiac rhythm to normal.

A similar approach has recently been applied to a neuronal network prepared from the hippocampal slice of the rat brain, by which one may gain a deeper insight into the pathogenesis and treatment of epilepsy [35].

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Appendix A. Glossary

Attractor: A structure in phase space, where trajectories are drawn to. Examples are stable fixed points, limit cycles, and strange attractors. *Bifurcation:* A brusque change in the number of states occupied by a deterministic system. Bifurcations denote the change in the type of dynamic motion when some parameter is altered. *Causality:* A principle in natural sciences. The weak principle of causality states that equivalent causes lead to the same effects. The strong principle of causality says that similar causes result in similar effects. Chaotic systems violate the strong principle of causality due to the sensitive dependence on initital conditions. *Chaos:* A type of dynamic behaviour sensitive to initial conditions and parameter changes, characterised by trajectories diverging exponentially with time, but restricted within a finite area in phase space. *Control parameter:* The difference of a parameter to a variable in an equation is that parameters are constant. They describe the strength of coupling between two or more variables or external forces. Control parameters can be altered to switch the system's behaviour from periodic to chaotic motion. *Delay differential system:* A description of a system where the current state depends on (or is a function of) the state at some time in the past. *Delay time:* The time by which the components of a reconstructed phase space vector are separated. It has to be chosen in such a way that the components are independent. This is often achieved by setting the delay time equal to the autocorrelation time of the signal (i.e., the first zero value of its autocorrelation function). *Deterministic:* A dynamical system in which equations of motion, variables, and parameters are specified exactly by some rule (e.g., by difference or differential equations). Through this information the motion of the system, however tangled, can be followed indefinitely in time. *Difference equation:* A set of rules that describe a system. The solution of a difference equation is discrete: i.e., $X_{i+1} = f(X_i)$, $i = 0,1,2....$ An example of a difference equation is the logistic equation: $X_{i+1} = aX_i(1 - X_i)$. A difference equation is often referred to as a ''map''. *Differential equation:* A set of rules that describe a system. The solution of a differential equation is continuous: i.e., $\dot{X}(t) = f(X(t))$, $t \in \mathrm{R}$. An example of a differential equation is the van der Pol oscillator. A differential equation is often referred to as a *flow*. *Dimension:* A measure of the amount of space an object occupies. There are only integer dimensions—point (0), line (1), surface (2), volume (3), etc.—in Euclidean space. A fractal dimension is also a measure of the ''space'' an object occupies; however, it is measured by quantifying the amount of space filled at different scales. The fractal dimension can be non-integer and may be greater than 3. *Dynamical systems theory:* A mathematical field that deals with the description of deterministic systems in terms of difference and differential equations as well as their solutions. *Feigenbaum diagram:* In systems described by difference equations (map) this is a plot of a set of control parameters against all points of the attractor for this parameter. In continuous systems (flows), the control parameter can be plotted against successive return points or points recorded at fixed phases etc. The Feigenbaum diagram is *self-similar*, which means that the magnification of a detail of the diagram looks like (or is similar to) the whole diagram. A Feigenbaum diagram is also *universal*: It it found in a large number of systems, which are described by a

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quadratic iteration function. Successive distances of two bifurcation points k_i have always the same ratio. In the limit $i \rightarrow \infty$ this ratio approaches the universal Feigenbaum constant $\delta = 4.669$. *Fractal dimension:* A measure of the extent to which a certain figure, motion or pattern fills n-dimensional space. Accordingly, a plane has the dimension of 2 and space has a dimension of three, whereas intermediate structures have a non-integral value of dimension between 2 and 3 or even higher. *Fixed point:* A point in phase space that attracts or repells a trajectory. There are at least four types of fixed points: When the system is attracted to an equilibrium state no matter where it starts from, the point is called *asymptotically stable* (also fixed point, node, or elliptic point). If the system will not approach the equilibrium state, the fixed point is *unstable* (also termed a "repeller"). If a fixed point provides a stable component and an unstable one, the system moves towards this point but then moves away from it (also an unstable fixed point, or saddle, or hyperbolic point). If there is a periodic solution around the fixed point, it is referred to as a vortex or an elliptic point. *Flow:* see *Differential equation.* *Intermittency:* At certain parameter values in the difference or differential equations, there are sequences of almost periodic behaviour interspersed with "chaotic" bursts (see Fig. 4). *Iteration:* A process in which the solution to an equation is fed back into the original equation as a new initial condition, which bears resemblance to a physiological feedback loop. *Limit cycle:* This is a periodic limit set for trajectories. A periodic attractor of a limit cycle may be embedded in more than two dimensions. The point about stable limit cycle solutions is that if a perturbation is imposed, the solution returns to the original periodic solution. The periodic behaviour is also independent of any initial data. *Lyapunov exponents:* A measure of the exponential divergence of a system in phase space, named after the Russian mathematician Alexander Lyapunov (1857–1918). A largest exponent of 0 indicates periodicity, whereas a positive exponents suggests a chaotic or stochastic system. *Map:* see *Difference equation.* *Non-linear:* A property of how two or more variables are coupled. In a broader sense, a system is non-linear if its output is not a linear function of the input. In contrast to the linear case the principle of superposition is not fulfilled. *Phase-locking:* The occurrence of periodic rhythms resulting from the interaction of two oscillators. Phase-locked rhythms are characterised by the ratio of two numbers reflecting the relative frequencies of both oscillators. In 4:3 phase-locking, during the time interval corresponding to the period of the rhythm, there are 4 cycles of one oscillation, and 3 of the other. In non-linear systems, the ratio of the two frequencies may change as parameters are changed. *Phase portrait:* A graphical technique that represents the relation between a point and any subsequent point in a time series. A lag 1 phase portrait is a plot of a value on one axis and the succeeding value of the time series on the other axis. Obviously, a lag 5 phase portrait is a plot of a value versus the 5th succeeding value. *Phase space:* An abstract coordinate system, which is defined by all independent variables of a system. The phase space of a plane pendulum is therefore two-dimensional, where the angle and the associated angular momentum are the independent coordinates. Lyapunov exponents and fractal dimensions are evaluated by phase–space coordinates. *Poincaré surface of section:* The intersection points of a trajectory moving from one side of a fixed plane in phase space to the other. If the system is periodic, the Poincaré section consists of distinct points (a single frequency with integer-multiple harmonics) or points lying on a closed curve (two or more incommensurable frequencies). For chaotic systems the surface of section is in general not completely filled but reveals some structure and order in chaos. Poincaré sections can be performed in more than two dimensions. *Predictability:* A measure of how far the state of a system can be predicted when the uncertainty in initial conditions is fixed. With linear growth of error the required time for increasing prediction grows linearly, too. In contrast to periodic systems, required time for increasing prediction time grows exponentially in the case of chaos (butterfly effect). *Quasi-periodic motion:* Periodic motion with two or more frequencies which are incommensurable (i.e., their ratio is not a number or a fraction of numbers). The motion of quasi-periodic systems is defined on a (n-dimensional) torus in phase space. *Random behaviour:* Behaviour that can never be predicted, and can only be described statistically such as the mean and standard deviation. *Return map:* A plot of a return value of the system (e.g., maximum angle of a pendulum) against the previous one. In a chaotic system, the signal may seem irregular, but the return map reveals some complicated structure far from randomness. *Self-similarity:* The property that a structure or pattern appears alike at any magnification. A natural example is the shape of a coastline. *Sensitive dependence on initial conditions:* In chaotic systems, tiny differences in initial conditions of two trajectories grow exponentially with time. A measure for divergence in phase space are Lyapunov exponents. Positive largest Lyapunov exponents emphasise the existence of chaotic dynamics or stochastic systems (see *Predictability*). *State space:* A coordinate system whose coordinates are all independent variables describing the behaviour of a particular system (see *Phase space*). *Strange attractor:* A strange attractor may be thought of as a complicatedly and complex-shaped surface in phase space, to which the system orbit is asymptotic in time and on which it wanders in a chaotic fashion. In general, the dimension of a strange attractor is non-integer. *Torus:* A periodic attractor in phase space, which is defined by at least two incommensurable frequencies: i.e., the two (or more) frequencies do not relate as the ratios of two (or more) numbers. With two frequencies, the system moves on the surface of a doughnut-like torus and densely fills the surface. *Trajectory:* The portrait of the behaviour of a system in state space or

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phase space over time.*Universality:* The prevalence of a Feigenbaum diagram (with associated Feigenbaum constant) in a great number of deterministic systems. Two deterministic systems, which reveal the Feigenbaum constant δ, both undergo a similar period-doubling bifurcation scenario.

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